AIAA JOURNAL
Vol. 38, No. 1, January 2000

Sensitivity Derivatives of Eigendata of One-Dimensional Structural Systems

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A general formulation based on the transfer matrix is presented to calculate the sensitivity derivatives of eigenvalue problems of one-dimensional structural systems. The method is equally applicable to any discrete or distributed system of one variable. The formulation is applied to determine the sensitivity derivatives of eigendata of a rotating helicopter blade and also to an optimization problem of a cantilever beam with frequency constraints. It is found that the computational efficiency of the present method is superior to those of the existing methods by an order of magnitude and that this efficacy improves with increasing numbers of design parameters and degrees of freedom.

Nomenclature

B_1, B_2	= section constants
b	= semichord of the blade
b_0	= semichord at the root
e	= distance between mass and elastic axis, positive when mass axis lies ahead of shear center
e_A	= distance between area centroid of tensile member and elastic axis, positive for centroid forward
e_0	= distance at root between elastic axis and axis about which blade is rotating, positive when elastic axis lies ahead
G	= shear modulus of elasticity
I_1, I_2	= bending moments of inertia about major and minor neutral axes, respectively
J	= torsional stiffness constant
k_m	= polar radius of gyration of cross-sectional mass about elastic axis, $k_{m1}^2 + k_{m2}^2$
k_{m1}, k_{m2}	= mass radii of gyration about major neutral axis and about an axis perpendicular to chord through the elastic axis, respectively
M_x, M_y, M_z	= resultant cross-sectional moments about x, y, and z directions, respectively
R	= span of the rotor
T	= tension in the blade
$[T]$	= transfer matrix
t	= thickness of cross section
V_y, V_z	= shear in y and z directions, respectively
v	= amplitude of simple harmonic linear displacement in the plane of rotation, positive toward leading edge
w	= amplitude of simple harmonic linear displacement normal to the plane of rotation, positive upward
X, Y, Z	= right-handed Cartesian coordinate system
β	= blade twist prior to deformation
ε	= slope of deflection curve in the plane of rotation
ϕ	= amplitude of simple harmonic torsional deformation, positive leading edge upward
ψ	= slope of deflection curve normal to plane of rotation
Ω	= angular velocity of rotation
ω	= frequency of vibration

Received 26 June 1997; revision received 24 July 1999; accepted for publication 31 July 1999.

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Introduction

SENSITIVITY derivatives are defined as the ratios of the variations in the system characteristics, such as natural frequencies or stress levels, to the variations in the design parameters, such as the mass or stiffness of the system. Automated optimization methods employ sensitivity derivatives to determine search directions towards optimum solutions. A close examination of optimization algorithms shows that the calculation of sensitivity derivatives constitutes a major portion of the cost and time. Additionally, there is growing interest in investigating optimization problems with large numbers of design variables and constraints that add significantly to the computational costs. Therefore, it is absolutely essential to develop efficient methods to calculate the sensitivity derivatives.

A comprehensive review on the subject of sensitivity analysis of discrete structural systems was presented in a paper by Adelman and Haftka.[1] Murthy and Haftka[2] summed up the methods available to calculate the sensitivity derivatives of general algebraic eigenvalue problems. Murthy and Lu[3] reviewed the sensitivity analysis methods of eigenvalue problems including those of periodic systems.

In this paper, a transfer matrix method is presented to calculate the sensitivity derivatives of eigenvalue problems of one-dimensional discrete, as well as distributed, systems. Optimization of structural systems with dynamic constraints requires the derivatives of eigendata. The literature dealing with the sensitivity analysis of distributed systems is relatively scarce, and a few references are cited by Adelman and Haftka.[1] The present formulation is well suited to both distributed and discrete systems. The differential equations of motion can be used directly in methods capable of dealing with distributed systems. This is particularly advantageous if the differential equations of motion are readily available because they can eliminate an intermediate step of generating a discrete model.

A large class of engineering systems can be modeled by one-dimensional systems such as simple beams, beams on elastic foundations, rotor or propeller blades, aircraft bulkheads, and turbine-generator shafts. The transfer matrix is ideally suited to treat such systems. The method was initially applied to analyze the torsional vibrations in shafts by Holzer.[4] An analogous method was applied by Myklestad[5] to determine the bending–torsion modes of beams that is commonly known as the Myklestad method. A complete and general development of the transfer matrix method may be found in the text by Pestel and Leckie[6] that also presents a catalog of transfer matrices for a large number of elastomechanical elements. The transfer matrix method was used to analyze helicopter blades by many researchers including Targoff,[7] Isakson and Eisley,[8] Murthy,[9] McDaniel and Murthy,[10] Murthy and Hammond,[11] and Shultz and Murthy.[12] The method is also widely used within the helicopter industry.

The present formulation is applied to calculate the sensitivity derivatives of eigendata of a helicopter blade. Differential equations of motion are derived by Houbolt and Brooks[13] for combined bending and torsion of twisted nonuniform rotating blades, and these are used in the application. The results obtained are compared with those obtained by a finite difference method for the validation of the present method. The computational efficiency in calculating the sensitivity derivatives is investigated by applying the formulation to a discrete beam model. Two general classes of methods are available to calculate the derivatives of discrete systems. One is known as the adjoint method, whereas the other is known as the direct method.[2,3] The adjoint method is used for comparing the computational efficiency of the present method. Next, the time efficiency is demonstrated in an optimization problem because the sensitivity derivatives are needed to determine the search directions toward the optimum solution. The program CONMIN[14] is used for the optimization. Note that in some cases the transfer matrix method can produce numerical difficulties. The problems usually arise when the higher natural frequencies are being calculated and when there are stiff intermediate elastic supports. These problems can usually be overcome by modifying the transfer matrix method[6] and by employing double precision. However, such difficulties are not encountered in the present applications.

Formulation

Consider a physical system described by linear differential equations of the form

$$\{Z(x)\}' = [A]\{Z(x)\} + \{f\} \tag{1}$$

where $\{Z(x)\}$ is the state vector of the system. The system matrix $[A]$ is assumed to be constant, and the vector $\{f\}$ represents the motion-independent applied forces.

The eigendata of the system can be computed by solving the homogeneous part of Eq. (1):

$$\{Z(x)\}' = [A]\{Z(x)\} \tag{2}$$

The solution of Eq. (2) can be written as

$$\{Z(x)\} = [T(x)]\{Z(0)\} \tag{3}$$

because for linear systems the degrees of freedom at x must always be a linear combination of the degrees of freedom at $x = 0$. Letting $x \to 0$ in Eq. (3) yields

$$[T(0)] = [I] \tag{4}$$

where $[I]$ is the identity matrix. Substituting Eq. (3) into Eq. (2) yields

$$([T(x)]' - [A(x)][T(x)])\{Z(0)\} = \{0\} \tag{5}$$

Because Eq. (5) must be valid for arbitrary values of $\{Z(0)\}$, it must be true that

$$[T(x)]' = [A(x)][T(x)] \tag{6}$$

The solution of Eq. (6) satisfying the initial conditions given by Eq. (4) provides the transfer matrix of the system.[9]

Equation (3) can be written for the whole length of the system as

$$\{Z(L)\} = [T(L)]\{Z(0)\} \tag{7}$$

where L is the length of the system. Equation (7) can be partitioned in terms of known and unknown elements of state vectors at the end points as

$$\left\{\begin{array}{c}\{Z(L)\}^k \\ \hline \{Z(L)\}^u\end{array}\right\} = \left[\begin{array}{c|c}[T(L)]^{ku} & [T(L)]^{kk} \\ \hline [T(L)]^{uu} & [T(L)]^{uk}\end{array}\right]\left\{\begin{array}{c}\{Z(0)\}^u \\ \hline \{Z(0)\}^k\end{array}\right\} \tag{8}$$

Note that the state vectors are split into known and unknown components to facilitate the application of the boundary conditions. The known quantities in the state vectors at 0 and L are, in general, not the same. It is simply a matter of notation that the known quantities are arranged at the top at L whereas they are at the bottom at 0. Extracting the top equation from Eq. (8) yields

$$\{Z(L)\}^k = [T(L)]^{ku}\{Z(0)\}^u + [T(L)]^{kk}\{Z(0)\}^k \tag{9}$$

If the boundary conditions of the problem are such that $\{Z(L)\}^k = \{Z(0)\}^k = \{0\}$, then Eq. (9) becomes

$$[T(L)]^{ku}\{Z(0)\}^u = \{0\} \tag{10}$$

(The formulation is applicable for other types of boundary conditions as well.) For a nontrivial solution of $\{Z(0)\}^u$ in Eq. (10) the determinant of the coefficient matrix must be equal to zero. Hence,

$$\det([T(L)]^{ku}) = 0 \tag{11}$$

The solution of Eq. (11) yields the eigenvalues ω of the system.

The eigenvectors of the problem can be obtained by expanding Eq. (10) as

$$\left[\begin{array}{c|ccc} T_{11}^{ku} & T_{12}^{ku} & \cdots & T_{1n}^{ku} \\ \hline T_{21}^{ku} & & & \\ \vdots & & [T^M] & \\ T_{n1}^{ku} & & & \end{array}\right]_{x=L} \left\{\begin{array}{c} z_1^u \\ \hline z_2^u \\ \vdots \\ z_n^u \end{array}\right\}_{x=0} = \{0\} \tag{12}$$

where the size of $[T^M]$ is $(n-1)\times(n-1)$. Assuming that the first element, z_1^u, is nonzero and arbitrarily assigning its value equal to 1, Eq. (12) can be rewritten as

$$\{Z(0)\}^u = \left\{\begin{array}{c} 1 \\ [T^M]^{-1}\{t_n\} \end{array}\right\} \tag{13}$$

where

$$\{t_n\}^T = -\lfloor T_{21}^{ku} \quad T_{31}^{ku} \quad \cdots \quad T_{n1}^{ku} \rfloor \tag{14}$$

If the first element, z_1^u, in Eq. (12) is zero, then $[T^M]$ may be singular and $\{Z(0)\}^u$ may not exist. In such a case the row corresponding to any nonzero element of $\{Z(0)\}^u$ has to be moved up to the top row. This process is very similar to the procedure employed in the calculation of mode shapes of a structural system by the transfer method.[9] Equation (3) can be written in partitioned form as

$$\{Z(x)\} = \left[\begin{array}{c|c}[T(x)]^{ku} & [T(x)]^{kk} \\ \hline [T(x)]^{uu} & [T(x)]^{uk}\end{array}\right]\left\{\begin{array}{c}\{Z(0)\}^u \\ \hline \{Z(0)\}^k = \{0\}\end{array}\right\} \tag{15}$$

Equation (15) can be simplified as

$$\{Z(x)\} = \begin{bmatrix}[T(x)]^{ku} \\ [T(x)]^{uu}\end{bmatrix}\{Z(0)\}^u \tag{16}$$

Substituting Eq. (13) into Eq. (16) yields

$$\{Z(x)\} = \begin{bmatrix}[T(x)]^{ku} \\ [T(x)]^{uu}\end{bmatrix}\left\{\begin{array}{c} 1 \\ [T^M]^{-1}\{t_n\} \end{array}\right\} \tag{17}$$

Eigenvectors can now be computed from Eq. (17).

The eigenvalues ω are computed from Eq. (11), and therefore $[T(L)]^{ku}$ has to be a function of ω. If p is a system parameter, then $[T(L)]^{ku}$ will also be a function of this parameter. Therefore, Eq. (11) can be written as

$$D = \det([T(\omega, p)]^{ku}) = 0 \tag{18}$$

Differentiating Eq. (18) yields

$$\mathrm{d}D = \left(\frac{\partial D}{\partial p}\right)\mathrm{d}p + \left(\frac{\partial D}{\partial \omega}\right)\mathrm{d}\omega = 0 \tag{19}$$

Equation (19) can be written as

$$\frac{\partial \omega}{\partial p} = -\frac{\partial D/\partial p}{\partial D/\partial \omega} \tag{20}$$

Similarly, Eq. (17) can be written as a function of ω and p as

$$\{Z(x,\omega,p)\} = \begin{bmatrix}[T(x,\omega,p)]^{ku} \\ [T(x,\omega,p)]^{uu}\end{bmatrix} \begin{Bmatrix} 1 \\ [T^M(\omega,p)]^{-1}\{t_n(\omega,p)\}\end{Bmatrix} \tag{21}$$

Differentiating Eq. (21) yields

$$\left\{\frac{\partial Z(x,\omega,p)}{\partial p}\right\} = \frac{\partial}{\partial p}\begin{bmatrix}[T(x,\omega,p)]^{ku} \\ [T(x,\omega,p)]^{uu}\end{bmatrix} \begin{Bmatrix} 1 \\ [T^M(\omega,p)]^{-1}\{t_n(\omega,p)\}\end{Bmatrix} + \begin{bmatrix}[T(x,\omega,p)]^{ku} \\ [T(x,\omega,p)]^{uu}\end{bmatrix} \frac{\partial}{\partial p}\begin{Bmatrix} 1 \\ [T^M(\omega,p)]^{-1}\{t_n(\omega,p)\}\end{Bmatrix} \tag{22}$$

Equations (20) and (22) provide the derivatives of the eigendata of the system.

Distributed Systems

If the system is nonuniform, the transfer matrix can be computed by solving Eq. (6) together with the initial conditions given by Eq. (4). Actually, the first-order matrix differential equation [Eq. (6)] can be integrated numerically starting with the initial conditions [Eq. (4)]. For uniform distributed systems the solution becomes

$$[T(x)] = e^{[A]x} \tag{23}$$

For sensitivity derivative calculations, as seen from Eqs. (22), (20), and (18), the derivatives of the transfer matrices with respect to the system parameters are needed. For computational efficiency there is a need to calculate these derivatives without recomputing the transfer matrices for small changes in the system parameters. This is accomplished by expanding the transfer matrices in a Taylor's series as

$$[T(x+\Delta x)] = [T(x)] + \Delta x[T(x)]' + (\Delta x^2/2)[T(x)]'' + \cdots \tag{24}$$

Differentiating Eq. (6) with respect to x yields

$$[T(x)]'' = [A(x)]'[T(x)] + [A(x)][T(x)]' \tag{25}$$

Substituting Eq. (6) into Eq. (25) yields

$$[T(x)]'' = ([A(x)]' + [A(x)]^2)[T(x)] \tag{26}$$

Substituting Eqs. (6) and (26) into Eq. (24) and retaining up to second-order terms in Δx yields

$$[T(x+\Delta x)] = ([I] + \Delta x[A(x)] + (\Delta x^2/2)([A(x)]' + [A(x)]^2))[T(x)] \tag{27}$$

Equation (27) can be written as

$$[T(x_{i+1})] = [C]_i[T(x_i)] \tag{28}$$

where

$$[C]_i = [I] + \Delta x[A(x_i)] + (\Delta x^2/2)([A(x_i)]' + [A(x_i)]^2)$$
$$x_i = x, \qquad x_{i+1} = x + \Delta x \tag{29}$$

Differentiating Eq. (28) with respect to p yields

$$\frac{\partial[T(x_{i+1})]}{\partial p} = \frac{\partial[C]_i}{\partial p}[T(x_i)] + [C]_i\frac{\partial[T(x_i)]}{\partial p} \tag{30}$$

where

$$\frac{\partial[C]_i}{\partial p} = \Delta x\frac{\partial[A(x_i)]}{\partial p} + \frac{\Delta x^2}{2}\frac{\partial}{\partial p}([A(x_i)]' + [A(x_i)]^2) \tag{31}$$

Equation (30) provides the recurrence relation to determine the derivatives of the transfer matrices with the starting matrices given by

$$[T(x_0)] = [I] \tag{32}$$

$$\frac{\partial[T(x_0)]}{\partial p} = [0] \tag{33}$$

Discrete Systems

The transfer matrix of a typical element (mass plus elastic element) can be obtained analytically, and the overall transfer matrix of a portion consisting of i elements is given by

$$[T]^i = [T]_i[T]_{i-1}\cdots[T]_2[T]_1, \qquad i = 1,\ldots,n \tag{34}$$

Equation (34) can be used to calculate the eigendata of the baseline system. For sensitivity derivative calculations the derivatives of the transfer matrices are needed. These can be computed by differentiating Eq. (34) as

$$\frac{\partial[T]^i}{\partial p} = \sum_{j=1}^{i}\left(\prod_{k=i}^{1}[U]_{jk}\right) \tag{35}$$

where

$$[U]_{jk} = [T]_k, \qquad \text{when} \qquad j \neq k \tag{36a}$$

$$[U]_{jk} = \frac{\partial}{\partial p}[T]_k, \qquad \text{when} \qquad j = k \tag{36b}$$

Because $[T]_k$ is known analytically for discrete systems, the derivatives required in Eq. (36b) can be computed readily.

For discrete structural systems the derivatives of eigendata can be computed by two conventional methods as described by Adelman and Haftka[1] and Murthy and Lu.[3] One method is known as the adjoint method, and the other is known as the direct method. In the conventional methods, the eigenvalue problems of discrete structural systems are formulated as

$$([A] - \omega_j[I])\{e_j\} = \{0\} \tag{37}$$

where $[A]$ is a constant system matrix, ω_j are eigenvalues of $[A]$, and $\{e_j\}$ are eigenvectors of the system. In the adjoint method the derivatives of eigendata are given by the following equations[3]:

$$\frac{\partial\omega_j}{\partial p} = \{q_j\}^T\frac{\partial[A]}{\partial p}\{e_j\} \tag{38}$$

$$\frac{\partial[E]}{\partial p} = [E][\varepsilon] \tag{39}$$

where the ijth element of the matrix $[\varepsilon]$ is given by

$$\varepsilon_{ij} = \frac{\{q_j\}^T(\partial[A]/\partial p)\{e_j\}}{\omega_j - \omega_i}, \qquad i \neq j \tag{40}$$

$$\varepsilon_{ii} = -\sum_{\substack{i=1\\ i\neq j}}^{n}\varepsilon_{ij}E_{ji} \tag{41}$$

The other quantities in Eqs. (38–41) are defined as follows:

p = design parameter
$\{e_j\}$ = jth eigenvector associated with eigenvalue ω_j
$\{q_j\}$ = jth adjoint eigenvector
$[E]$ = eigenvector matrix

In the direct method the derivatives of eigendata can be expressed as[3]

$$\frac{\partial}{\partial p}\begin{Bmatrix}\omega_j \\ \{e_j\}_i\end{Bmatrix} = [B]^{-1}\frac{\partial[A]}{\partial p}\{e_j\} \tag{42}$$

where $[B] = [\{e_j\}, (\omega_j[I] - [A])]$ with the ith column omitted and $\{e_j\}_i = \{e_j\}$ with ith element omitted.

A special case of the adjoint method corresponds to structural vibration systems where the mass and stiffness matrices are symmetric. A method developed by Fox and Kapoor[15] makes use of this symmetry that results in an expression for the derivative of the eigenvalue that depends only on the one eigenvector corresponding to the eigenvalue. For an eigenvalue problem of the form

$$[K]\{x\} = \omega[M]\{x\} \tag{43}$$

the derivative of the eigenvalue with respect to a design parameter is given by

$$\frac{\partial \omega_i}{\partial p_j} = \{x_i\}^T\left(\left[\frac{\partial K}{\partial p_j}\right] - \omega_i\left[\frac{\partial M}{\partial p_j}\right]\right)\{x_i\} \tag{44}$$

where the eigenvectors have been normalized such that

$$\{x_i\}^T[M]\{x_i\} = 1 \tag{45}$$

Analytical Example

A simple analytical example of a beam problem is included here for a clearer understanding of the formulation. The eigendata of a uniform beam in bending vibration are governed by

$$EIw'''' - \omega^2 mw = 0 \tag{46}$$

Equation (46) in state vector form can be written as

$$\{Z(x)\}' = [A]\{Z(x)\} \tag{47}$$

where

$$\{Z(x)\}^T = \lfloor w \quad \psi \quad M \quad -V \rfloor$$

and w is deflection, $\psi = w'$ is slope, $M = EIw''$ is bending moment, $V = -EIw'''$ is shear force, and

$$[A] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/EI & 0 \\ 0 & 0 & 0 & 1 \\ \omega^2 m & 0 & 0 & 0 \end{bmatrix}$$

The transfer matrix of the system is given by

$$[T(x)]' = [A][T(x)] \tag{48}$$

with $[T(0)] = [I]$.

The state vectors at the end points of a cantilever beam (0 and L) can be related as

$$\{Z(L)\} = [T]\{Z(0)\} \tag{49}$$

Equation (49) can be partitioned in terms of known and unknown quantities of the state vector as

$$\left\{\begin{array}{c} 0 \\ 0 \\ \hline M \\ -V \end{array}\right\}_{x=L} = \left[\begin{array}{cc|cc} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ \hline T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{array}\right]_{x=L} \left\{\begin{array}{c} w \\ y \\ \hline 0 \\ 0 \end{array}\right\}_{x=0} \tag{50}$$

($x = L$ is the fixed end where $w = \psi = 0$, and $x = 0$ is the free end where $M = V = 0$). The frequency determinant is given by

$$D = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} = T_{11}T_{22} - T_{12}T_{21} = 0 \tag{51}$$

The sensitivity derivative of a natural frequency with respect to a parameter p is given by

$$\frac{\partial \omega}{\partial p} = -\frac{\partial D/\partial p}{\partial D/\partial \omega} = \frac{(\partial/\partial p)(T_{11}T_{22} - T_{12}T_{21})}{(\partial/\partial \omega)(T_{11}T_{22} - T_{12}T_{21})} \tag{52}$$

The recurrence relations for the calculation of the transfer matrix and its derivatives with respect to a parameter p are given by (here p can be EI, m, or ω)

$$[T(x_{i+1})] = [C][T(x_i)], \qquad i = 0, \ldots, n \tag{53}$$

$$\frac{\partial[T(x_{i+1})]}{\partial p} = \frac{\partial[C]}{\partial p}[T(x_i)] + [C]\frac{\partial[T(x_i)]}{\partial p} \tag{54}$$

where

$$[C] = [I] + \Delta x[A] + (\Delta x^2/2)[A]^2, \qquad \Delta x = L/n$$

The starting values for the recurrence relations are given by

$$[T(x_0)] = [I] \tag{55}$$

$$\frac{\partial[T(x_0)]}{\partial p} = [0] \tag{56}$$

The preceding recurrence relations can be used to calculate the elements of the transfer matrices and their derivatives that appear in Eqs. (51) and (52). Initial baseline natural frequencies can be computed from Eq. (51) employing a frequency scanning method. Similarly, the derivatives of eigenvectors can be calculated using Eq. (22).

Applications

The method developed here is applied to determine the derivatives of eigendata of a distributed model of a nonuniform, twisted, rotating cantilever helicopter blade with coupled bending and torsional degrees of freedom. Most of the beam problems of interest will be special subcases of this problem. For simplicity actual numerical calculations are carried out for a blade with uniform data. Next, the method is applied to a simple discrete cantilever beam to investigate the computational efficiency when compared to existing methods. Then, an optimization problem is examined to determine the final computational efficiency.

Distributed Systems

The blade differential equations of motion for combined bending and torsion of a twisted nonuniform helicopter blade are derived by Houbolt and Brooks[13] and are given next for simple harmonic free vibration with frequency ω:

$$\begin{aligned} &-\left\{\left[GJ + Tk_A^2 + EB_1(\beta')^2\right]\phi' - EB_2\beta'(v''\cos\beta + w''\sin\beta)\right\}' \\ &\quad + Te_A(v''\sin\beta - w''\cos\beta) + \Omega^2 mxe(-v'\sin\beta + w'\cos\beta) \\ &\quad + \Omega^2 me\sin\beta v + \Omega^2 m\left[(k_{m2}^2 - k_{m1}^2)\cos 2\beta + ee_0\cos\beta\right]\phi \\ &\quad - \omega^2 mk_m^2\phi + \omega^2 me(v\sin\beta - w\cos\beta) = 0 \end{aligned} \tag{57}$$

$$\begin{aligned} &\left[(EI_1\cos^2\beta + EI_2\sin^2\beta)w'' + (EI_2 - EI_1)\sin\beta\cos\beta v''\right. \\ &\quad \left.- Te_A\phi\cos\beta - EB_2\beta'\phi'\sin\beta\right]'' - (Tw')' - (\Omega^2 mxe\phi\cos\beta)' \\ &\quad - \omega^2 m(w + e\phi\cos\beta) = 0 \end{aligned} \tag{58}$$

$$\begin{aligned} &\left[(EI_2 - EI_1)\sin\beta\cos\beta w'' + (EI_1\sin^2\beta + EI_2\cos^2\beta)v''\right. \\ &\quad \left.+ Te_A\phi\sin\beta - EB_2\beta'\phi'\cos\beta\right]'' - (Tv')' + (\Omega^2 mxe\phi\sin\beta)' \\ &\quad + \Omega^2 me\phi\sin\beta - \omega^2 m(v - e\phi\sin\beta) - \Omega^2 mv = 0 \end{aligned} \tag{59}$$

For determination of the transfer matrix it is desirable to reduce the governing differential equations of motion to a set of first-order differential equations. The resulting equations are

$$\frac{d\bar{w}}{d\bar{x}} = \bar{\psi} \tag{60}$$

$$\frac{d\bar{v}}{d\bar{x}} = \bar{\varepsilon} \tag{61}$$

$$\begin{aligned} \frac{d\bar{\psi}}{d\bar{x}} &= c_{31}\bar{M}_x + c_{32}\bar{M}_y + c_{33}\bar{M}_z + c_{31}\bar{T}\bar{e}_A\sin\beta\bar{\varepsilon} \\ &\quad - c_{31}\bar{T}\bar{e}_A\cos\beta\bar{\psi} + (c_{32}\cos\beta - c_{33}\sin\beta)\bar{T}\bar{e}_A\bar{\phi} \end{aligned} \tag{62}$$

$$\frac{d\bar{\varepsilon}}{d\bar{x}} = c_{21}\bar{M}_x + c_{22}\bar{M}_y + c_{23}\bar{M}_z + c_{21}\bar{T}\bar{e}_A \sin\beta\bar{\varepsilon} - c_{21}\bar{T}\bar{e}_A\cos\beta\bar{\psi} + (c_{22}\cos\beta - c_{23}\sin\beta)\bar{T}\bar{e}_A\bar{\phi} \tag{63}$$

$$\frac{d\bar{\phi}}{d\bar{x}} = c_{11}\bar{M}_x + c_{12}\bar{M}_y + c_{13}\bar{M}_z + c_{11}\bar{T}\bar{e}_A \sin\beta\bar{\varepsilon} - c_{11}\bar{T}\bar{e}_A\cos\beta\bar{\psi} + (c_{12}\cos\beta - c_{13}\sin\beta)\bar{T}\bar{e}_A\bar{\phi} \tag{64}$$

$$\begin{aligned}\frac{d\bar{M}_x}{d\bar{x}} = {} & \left[\bar{\Omega}^2\left(\frac{m}{m_0}\right)\bar{x}\sin\beta(\bar{e}_A - \bar{e}) - \bar{T}\bar{e}'_A\sin\beta - \bar{T}\bar{e}_A\beta'\cos\beta\right]\bar{\varepsilon} \\ & + \left[\bar{\Omega}^2\left(\frac{m}{m_0}\right)\bar{x}\cos\beta(\bar{e} - \bar{e}_A) - \bar{T}\bar{e}'_A\cos\beta - \bar{T}\bar{e}_A\beta'\sin\beta\right]\bar{\psi} \\ & + \left[(\bar{\Omega}^2 + \bar{\omega}^2)\left(\frac{m}{m_0}\right)\bar{e}\sin\beta\right]\bar{v} - \left[\bar{\omega}^2\left(\frac{m}{m_0}\right)\bar{e}\cos\beta\right]\bar{w} \\ & + \left[\bar{\Omega}^2\left(\frac{m}{m_0}\right)\left\{(\bar{k}^2_{m2} - \bar{k}^2_{m1})\cos 2\beta + \bar{e}\bar{e}_0\cos\beta\right\} - \bar{\omega}^2\left(\frac{m}{m_0}\right)\bar{k}^2_m\right]\bar{\phi}\end{aligned} \tag{65}$$

$$\frac{d\bar{M}_z}{d\bar{x}} = \bar{T}\bar{\varepsilon} - \bar{V}_y - \bar{\Omega}^2\left(\frac{m}{m_0}\right)\bar{e}\bar{x}\sin\beta\bar{\phi} \tag{66}$$

$$\frac{d\bar{M}_y}{d\bar{x}} = \bar{T}\bar{\psi} - \bar{V}_z + \bar{\Omega}^2\left(\frac{m}{m_0}\right)\bar{e}\bar{x}\cos\beta\bar{\phi} \tag{67}$$

$$\frac{-d\bar{V}_y}{d\bar{x}} = (\bar{\Omega}^2 + \bar{\omega}^2)\left(\frac{m}{m_0}\right)\bar{v} - (\bar{\Omega}^2 + \bar{\omega}^2)\left(\frac{m}{m_0}\right)\bar{e}\sin\beta\bar{\phi} \tag{68}$$

$$\frac{-d\bar{V}_z}{d\bar{x}} = \bar{\omega}^2\left(\frac{m}{m_0}\right)\bar{w} - \bar{\omega}^2\left(\frac{m}{m_0}\right)\bar{e}\cos\beta\bar{\phi} \tag{69}$$

The nondimensional state vector and coefficients appearing in the preceding first-order differential equations are defined in the Appendix. Equations (60–69) define the elements of the coefficient matrix $[A]$ in Eq. (6) for evaluation of the transfer matrix. Once the transfer matrix is determined the frequency equation can be obtained in terms of the elements of the transfer matrix for a specified set of boundary conditions.

The frequency determinant corresponding to a given set of boundary conditions can be determined in the following manner. For cantilever blades the boundary conditions are, at the fixed end,

$$\bar{w} = \bar{v} = \bar{\psi} = \bar{\varepsilon} = \bar{\phi} = 0 \tag{70}$$

and at the free end,

$$\bar{M}_x = \bar{M}_y = \bar{M}_z = \bar{V}_y = \bar{V}_z = 0 \tag{71}$$

By definition of the transfer matrix one can write

$$\{Z\}_{\bar{x}=1} = [T_{ij}]\{Z\}_{\bar{x}=0} \tag{72}$$

where

$$\{Z\}^T = \lfloor \bar{w} \quad \bar{v} \quad \bar{\psi} \quad \bar{\varepsilon} \quad \bar{\phi} \quad \bar{M}_x \quad \bar{M}_z \quad \bar{M}_y \quad -\bar{V}_y \quad -\bar{V}_z \rfloor$$

Let $\bar{x} = 0$ be defined as the fixed end and $\bar{x} = 1$ the free end. By the substitution of the boundary conditions given by Eqs. (70) and (71) into Eq. (72) the following frequency equation results:

$$\begin{vmatrix} T_{66} & T_{67} & T_{68} & T_{69} & T_{6\,10} \\ T_{76} & T_{77} & T_{78} & T_{79} & T_{7\,10} \\ T_{86} & T_{87} & T_{88} & T_{89} & T_{8\,10} \\ T_{96} & T_{97} & T_{98} & T_{99} & T_{9\,10} \\ T_{10\,6} & T_{10\,7} & T_{10\,8} & T_{10\,9} & T_{10\,10} \end{vmatrix} = 0 \tag{73}$$

Once the natural frequencies are determined by solving Eq. (73), the associated mode shapes can be obtained from the following equations:

$$\begin{Bmatrix} \bar{w}(\bar{x}) \\ \bar{v}(\bar{x}) \\ \bar{\phi}(\bar{x}) \end{Bmatrix} = \begin{bmatrix} T_{16}(\bar{x}) & T_{17}(\bar{x}) & T_{18}(\bar{x}) & T_{19}(\bar{x}) \\ T_{26}(\bar{x}) & T_{27}(\bar{x}) & T_{28}(\bar{x}) & T_{29}(\bar{x}) \\ T_{56}(\bar{x}) & T_{57}(\bar{x}) & T_{58}(\bar{x}) & T_{59}(\bar{x}) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} - \begin{Bmatrix} T_{1\,10}(\bar{x}) \\ T_{2\,10}(\bar{x}) \\ T_{5\,10}(\bar{x}) \end{Bmatrix} \tag{74}$$

where

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} T_{66} & T_{67} & T_{68} & T_{69} \\ T_{76} & T_{77} & T_{78} & T_{79} \\ T_{86} & T_{87} & T_{88} & T_{89} \\ T_{96} & T_{97} & T_{98} & T_{99} \end{bmatrix}^{-1} \begin{Bmatrix} T_{6\,10} \\ T_{7\,10} \\ T_{8\,10} \\ T_{9\,10} \end{Bmatrix} \tag{75}$$

and the elements T_{ij} are defined in Eq. (72).

The eigendata of a baseline rotor blade are computed from Eqs. (73–75). The elements of the transfer matrix required in these equations are computed numerically from Eqs. (4) and (6). This transfer matrix corresponds to the baseline system and is computed only once. The derivatives of eigendata are computed using Eqs. (20) and (22). These equations need the derivatives of the elements of the transfer matrix that are computed from Eq. (30).

A uniform rotor blade example is considered for numerical calculations, and the data are $R = 260$ in., $\beta = -10$ deg linear pretwist, $m = 1.5 \times 10^{-3}$ slug/in., $EI_1 = 3.0 \times 10^7$ lb-in.^{2}, $EI_2 = 1.0 \times 10^9$ lb-in.^{2}, $GJ = 2.0 \times 10^7$ lb-in.^{2}, $k^2_{m1} = 0.5997$ in.^{2}, $k^2_{m2} = 26.6667$ in.^{2}, $e = -0.6$ in., $b = 6.0$ in., $\Omega = 360$ rpm, and $k_A = B_1 = B_2 = e_A = e_0 = 0$.

The baseline natural frequencies and the associated predominant modeshapes are given in Table 1 for the first four modes. The derivatives of the baseline frequencies in Table 1 are calculated and presented in Table 2. The results obtained are compared with those obtained using a finite difference method. Similarly the derivatives of mode shapes with respect to all of the design parameters are computed. For the sake of brevity, the derivative of select eigenvectors of only the first mode is presented in Table 3. Again the results are compared with those obtained using a finite difference method.

Discrete Systems

To investigate the CPU time comparisons between the present and existing methods, a discretized cantilever beam model is employed. The discretized model consists of five elements with the following uniform properties: $L = 40$ in., $m = 0.000125$ lb/in., and $EI = 25{,}000$ lb-in.^{2}. A diagonal lumped mass matrix and a cubic polynomial based finite element stiffness matrix are employed for comparison with the present transfer matrix based formulation. The baseline eigendata given by both the methods agree as expected and the first three natural frequencies are $\omega_1 = 25.8799$, $\omega_2 = 165.2513$, and $\omega_3 = 468.2280$. For comparison of derivatives of eigendata the adjoint method is selected because it is found to be efficient for the present problem. The results including the CPU times are presented in Tables 4 and 5.

Table 1 Baseline natural frequencies and predominant normalized mode shapes

Axial position x	First mode ω_1 = 40.0429, rad/s $\{w_1\}$	Second mode ω_2 = 45.3187, rad/s $\{v_2\}$	Third mode ω_3 = 105.6492, rad/s $\{w_3\}$	Fourth mode ω_4 = 138.7851, rad/s $\{\phi_4\}$
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.0453	0.0188	−0.0982	0.1557
0.2	0.1330	0.0695	−0.2728	0.3076
0.3	0.2338	0.1454	−0.4347	0.4520
0.4	0.3392	0.2406	−0.5386	0.5855
0.5	0.4468	0.3504	−0.5529	0.7048
0.6	0.5558	0.4709	−0.4547	0.8071
0.7	0.6659	0.5985	−0.2322	0.8896
0.8	0.7769	0.7306	0.1083	0.9503
0.9	0.8883	0.8649	0.5357	0.9875
1.0	1.0000	1.0000	1.0000	1.0000

Table 2 Comparisons of derivatives of eigenvalues with respect to design parameters

Mode number	Transfer matrix method				Finite difference method			
	$\partial\omega/\partial m$	$\partial\omega/\partial EI_1$	$\partial\omega/\partial EI_2$	$\partial\omega/\partial GJ$	$\partial\omega/\partial m$	$\partial\omega/\partial EI_1$	$\partial\omega/\partial EI_2$	$\partial\omega/\partial GJ$
1	−1.279	1.25	0.0273	$2.42E-5$	−1.270	1.249	0.0263	$2.40E-5$
2	−19.81	0.0423	19.78	$3.71E-6$	−19.66	0.0421	19.73	$3.68E-6$
3	−11.59	10.36	0.3276	0.9105	−11.50	10.35	0.3282	0.8916
4	−64.24	0.1780	$9.32E-3$	64.06	−63.74	0.1780	$9.34E-3$	63.92
5	−42.48	41.19	1.029	0.2650	−42.12	41.13	1.011	0.2619

Table 3 Comparisons of derivatives of eigenvectors

Axial position x	Transfer matrix method		Finite element method	
	$\partial\{w\}/\partial m$	$\partial\{v\}/\partial m$	$\partial\{w\}/\partial m$	$\partial\{v\}/\partial m$
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.0131	0.0025	0.0130	0.0026
0.2	0.0243	0.0090	0.0241	0.0093
0.3	0.0276	0.0185	0.0274	0.0190
0.4	0.0265	0.0301	0.0262	0.0310
0.5	0.0232	0.0434	0.0229	0.0446
0.6	0.0190	0.0578	0.0187	0.0595
0.7	0.0143	0.0730	0.0141	0.0752
0.8	0.0095	0.0888	0.0093	0.0914
0.9	0.0047	0.1047	0.0046	0.1078
1.0	0.0000	0.1207	0.0000	0.1243

Table 4 Comparison of CPU time: eigenvalue derivatives of three natural frequencies

Number of stations	One-parameter CPU time $\partial\omega_i/\partial m_1$, $i = 1, 2, 3$ Transfer matrix method	One-parameter CPU time Adjoint method	Two-parameter CPU time $\partial\omega_i/\partial m_1$ and $\partial\omega_i/\partial m_2$, $i = 1, 2, 3$ Transfer matrix method	Two-parameter CPU time Adjoint method
10	0.15	0.21	0.17	0.23
30	0.36	0.44	0.39	0.47
60	0.54	2.10	0.57	2.16
90	0.63	6.17	0.69	6.25
120	0.96	14.00	1.08	14.15
150	1.14	26.30	1.26	26.51

Table 5 Comparison of CPU time: eigenvector derivatives of first mode for cantilever beam

Number of stations	One-parameter CPU time $\partial\{w_1\}/\partial m_1$ Transfer matrix method	One-parameter CPU time Adjoint method	Two-parameter CPU time $\partial\{w_1\}/\partial m_1$ and $\partial\{w_1\}/\partial m_2$ Transfer matrix method	Two-parameter CPU time Adjoint method
10	0.06	0.10	0.07	0.11
30	0.13	1.41	0.18	1.48
60	0.21	9.81	0.30	10.22
90	0.24	31.48	0.35	32.76
120	0.35	71.37	0.49	74.40
150	0.42	137.13	0.58	143.02

Discussion of Results

Distributed Systems

The main objective here is to develop and validate the transfer matrix method for calculating the sensitivity derivatives of one-dimensional structural systems. The method is very appealing because it is applicable to distributed systems. This eliminates an intermediate step of generating a suitable discrete model representing the distributed system. For instance, the differential equations of motion governing the dynamics of a nonuniform and pretwisted helicopter blade with coupled bending and torsional degrees of freedom are readily available in the literature.[13] The present method is directly applied to this model, and the results are compared with those obtained using a finite difference method. Baseline natural frequencies and the associated components of the predominant normalized modes are presented in Table 1. The baseline data are provided here just to define the characteristics of the blade model. This will facilitate future comparisons of the present results. Comparisons of the eigenvalue and the associated eigenvector derivatives are presented in Tables 2 and 3, respectively. These comparisons clearly validate the formulation and its application to a fairly general one-dimensional structural problem. It is not intended here to provide any guidelines to a real-life rotor optimization problem because it involves several dynamic, aeroelastic, and aeromechanical constraints based on the rotor configurations.[16] The design parameters selected for this distributed example are m, EI_1, EI_2, and GJ. The sensitivity derivatives with respect to these parameters are computed by a variation of the distribution as follows. Consider the mass variation as $m_{\text{modified}}(x) = m(x) + pm(x)$. Then the derivative of the natural frequency with respect to the mass variation is computed as $\partial\omega/\partial p$. Other variations including concentrated masses can be handled in the formulation. Similarly, the other parameters of the problem are handled in the same fashion. In the results computed by the finite difference method the design parameter p is increased to $p + \Delta p$, where $\Delta p = 0.001$, to calculate the derivatives. The main objective here is to validate the present formulation and, therefore, no studies are performed to examine the effects of either the finite difference increment or the central difference technique, and the derivatives are calculated as $\partial\omega/\partial p = [\omega(p + \Delta p) - \omega(p)]/\Delta p$.

Discrete Systems

The main objective here is to investigate the computational efficiency of the present method relative to the existing methods. It is found that a significant additional effort is needed to apply the existing methods, such as the adjoint and direct methods, to calculate the sensitivity derivatives of the rotor blade problem described by Eqs. (57–59). This is due to the need for generating a discrete or a finite element model corresponding to Eqs. (57–59). This is one of the primary advantages of the present method, it can be formulated directly from the equations of motion. Therefore, a simple nonrotating beam is considered for comparison with the existing methods because a discrete model can easily be generated for this problem.

Murthy and Haftka[2] performed relative computational cost studies of direct and adjoint methods and reported the following results for the number of operations:

1) For the calculation of first derivatives of eigenvalues, the number of operations with the adjoint method was $j(\frac{7}{2}n^2 + kmn^2)$ and with the direct method was $j[n^3/3 + (k+1)mn^2]$.

2) For the calculation of first derivatives of eigenvalues and eigenvectors, the number of operations with the adjoint method was $\frac{7}{2}n^3 + jmn^2(k+2)$ and with the direct method was $j[n^3/3 + (k+1)mn^2]$, where j is the number of eigenvalues of interest, m the number of design parameters, n the matrix size, and k the scarcity factor of derivative of $[A]$ ($k = 1$ for full matrix).

From the computation cost results, it is concluded that for $n > 10$ the adjoint method will be more efficient than the direct method for calculations of derivatives of eigenvalues. Therefore, the adjoint method is selected for comparison with the present method. The computational times required for the present method are compared with the adjoint method in Tables 4 and 5. The results clearly indicate that the present method significantly reduces the computation times required for calculations of the derivatives of eigenvalues. The efficiency increases significantly with the increasing number of degrees of freedom. However, Nelson's[17] method, which is a simplified form of the adjoint method, may be better for calculation of the eigenvector derivatives than a general adjoint method. A closer look at the calculation of the derivatives of eigenvectors by the present method and Nelson's method indicates that an inversion of a 2×2 matrix would be needed in the present method (independent of n, the number of segments), whereas an inversion of $(n-1) \times (n-1)$ [or solution of $(n-1)$ simultaneous equations] would be needed for the latter. This formulation difference will provide a clear edge to the present method over the existing methods.

Optimization Problem

Optimization programs such as CONMIN[14] need sensitivity derivatives of objective and constraint functions with respect to the design parameters. A closer examination of the optimization algorithms shows that the calculation of these derivatives constitutes a major portion of the cost and time. If the derivative information is not provided by the user, the programs usually compute these by a finite difference method. The computational efficiency of the present method of calculating the sensitivity derivatives of an optimization problem is investigated here.

The problem considered here is a weight minimization of a discrete cantilever beam with a tip mass subjected to a frequency constraint. This problem has been studied by several researchers including Turner,[18] Kahn and Willmert,[19] and Peters et al.[20] This problem is very convenient for comparison.

The beam is divided into four equal lengths. The weight minimization essentially becomes a problem of minimizing the sum of the four individual areas. The specified fundamental frequency is given as 17.52 rad/s, and the other data are modulus of elasticity = 10.3×10^6 psi; mass density = 2.5×10^{-4} lb-s^2/in.^{4}; radius of gyration $(A_1) = 2.0$ in., $(A_2) = 1.5$ in., $(A_3) = 1.0$ in., and $(A_4) = 0.5$ in.; tip mass = 1.0 lb-s^2/in.; and length of each segment = 60 in. A starting solution is chosen to be $A_1 = 200$ in.^{2}, $A_2 = 150$ in.^{2}, $A_3 = 60$ in.^{2}, and $A_4 = 35$ in.^{2}. The optimization program CONMIN is used for the analysis. Results are shown in Table 6, and it can be seen that these clearly validate the present application of the formulation.

To investigate the computational efficiency of the present method the beam model was changed to incorporate any number of elements. The number of beam elements defines the number of decision variables of the optimization problem, and the program will vary each elemental area independently until an optimum solution is achieved. In Table 7, the results of the present formulation are compared with those obtained using an adjoint type of formulation and by a finite difference method. Actually, Fox and Kapoor's method,[15] which can be considered as a special case of the adjoint type of method with symmetric mass and stiffness matrices, is employed in the calculation of the results presented in the second column of Table 7. A built-in finite difference method in CONMIN[14] is used for the calculation of the results in the third column. It can be seen from the results of Table 7 that the present formulation can significantly cut down the computational times. The efficiency increases rapidly with increasing numbers of degrees of freedom.

Table 6 Comparison of optimization results

Decision variable	Turner[18]	Kahn and Willmert[19]	Peters et al.[20]	Present method
Area, in.^{2}				
A_1	136.81	136.63	134.60	137.73
A_2	118.73	118.70	116.58	114.13
A_3	83.591	83.586	82.734	82.161
A_4	34.427	34.608	34.898	34.874
Beam weight, lb	2243.0	2242.9	2214.41	2214.93

Table 7 Comparisons of CPU times for optimization problem

Number of beam elements	Present transfer matrix method	Adjoint method	Finite difference method
4	0.27	0.50	0.31
8	0.44	1.40	2.26
12	0.50	4.56	7.09
16	1.05	12.03	22.47
24	1.13	66.41	110.62

Conclusion

This paper presents a transfer-matrix-based method to calculate the sensitivity derivatives of eigendata of one-dimensional structural systems. The method is capable of dealing with both distributed and discrete systems. Methods capable of dealing with distributed systems are sometimes very appealing because they eliminate an intermediate step of generating the discrete model. The present formulation provides significant computational efficiency over existing methods in calculating the sensitivity derivatives, which ultimately reflects in the final optimization. The efficiency increases with increasing number of degrees of freedom and design parameters.

Appendix: Nondimensional Coefficients

The nondimensional elements of the state vector appearing in Eqs. (60–69) are (where the subscript zero refers to reference values at the root)

$$\bar{w} = w/b_0, \qquad \bar{v} = v/b_0, \qquad \bar{\psi} = \psi R/b_0$$

$$\bar{\phi} = \phi R/b_0, \qquad \bar{\varepsilon} = \varepsilon R/b_0$$

$$\bar{M}_x = M_x R^3/(EI_{10} b_0^2), \qquad \bar{M}_y = M_y R^2/(EI_{10} b_0)$$

$$\bar{M}_z = M_z R^2/(EI_{10} b_0)$$

$$\bar{V}_y = V_y R^3/(EI_{10} b_0), \qquad \bar{V}_z = V_z R^3/(EI_{10} b_0)$$

$$\bar{x} = x/R, \qquad \bar{k}_A^2 = k_A^2/b_0^2, \qquad \bar{\beta} = \beta$$

$$\bar{e} = e/b_0, \qquad \bar{e}_A = e_A/b_0, \qquad \bar{e}_0 = e_A/b_0$$

$$\bar{k}_{m1}^2 = k_{m1}^2/b_0^2, \qquad \bar{k}_{m2}^2 = k_{m2}^2/b_0^2, \qquad \bar{k}_m^2 = k_m^2/b_0^2$$

$$\bar{\Omega}^2 = \Omega^2 m_0 R^4/EI_{10}, \qquad \bar{\omega}^2 = \omega^2 m_0 R^4/EI_{10}$$

$$c_{11} = (a_{11}a_{32} - a_{23}a_{32})/D, \qquad c_{12} = (a_{13}a_{32} - a_{12}a_{33})/D$$

$$c_{13} = (a_{12}a_{23} - a_{13}a_{22})/D$$

$$c_{21} = (a_{23}a_{31} - a_{21}a_{33})/D, \qquad c_{22} = (a_{11}a_{33} - a_{13}a_{31})/D$$

$$c_{23} = (a_{13}a_{21} - a_{11}a_{33})/D$$

$$c_{31} = (a_{21}a_{32} - a_{22}a_{31})/D, \qquad c_{32} = (a_{12}a_{31} - a_{11}a_{32})/D,$$

$$c_{33} = (a_{11}a_{22} - a_{12}a_{21})/D, \qquad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$a_{11} = (R/b_0)^2(GJ/EI_{10}) + \bar{T}\bar{k}_A^2 + EB_1(\beta')^2/(EI_{10}b_0^2)$$

$$a_{12} = (-EB_2\beta'/EI_{10}b_0^2)\cos\beta = a_{31}$$

$$a_{13} = (-EB_2\beta'/EI_{10}b_0^2)\sin\beta = a_{21}$$

$$a_{22} = [(EI_2/EI_{10}) - (EI_1/EI_{10})]\sin\beta\cos\beta = a_{33}$$

$$a_{23} = (EI_1/EI_{10})\cos^2\beta + (EI_2/EI_{10})\sin^2\beta$$

$$a_{32} = (EI_1/EI_{10})\sin^2\beta + (EI_2/EI_{10})\cos^2\beta$$

$$\bar{T} = \frac{TR^2}{EI_{10}} = \bar{\Omega}^2 \int_{\bar{x}}^{1} \frac{m}{m_0}\bar{x}\,\mathrm{d}\bar{x}$$

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Associate Editor